

A new basis for eigenmodes on the sphere

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2004 J. Phys. A: Math. Gen. 37 205

(<http://iopscience.iop.org/0305-4470/37/1/014>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.89

The article was downloaded on 02/06/2010 at 17:26

Please note that [terms and conditions apply](#).

A new basis for eigenmodes on the sphere

M Lachièze-Rey

Service d'Astrophysique, CE Saclay 91191, Gif sur Yvette cedex, France

E-mail: marclr@cea.fr

Received 9 October 2003

Published 10 December 2003

Online at stacks.iop.org/JPhysA/37/205 (DOI: 10.1088/0305-4470/37/1/014)

Abstract

The usual spherical harmonics $Y_{\ell m}$ form a basis of the vector space \mathcal{V}^ℓ (of dimension $2\ell + 1$) of the eigenfunctions of the Laplacian on the sphere, with eigenvalue $\lambda_\ell = -\ell(\ell + 1)$. Here we show the existence of a different basis Φ_j^ℓ for \mathcal{V}^ℓ , where $\Phi_j^\ell(X) \equiv (X \cdot N_j)^\ell$, the ℓ th power of the scalar product of the current point with a specific null vector N_j . We give explicitly the transformation properties between the two bases. The simplicity of calculations in the new basis allows easy manipulations of the harmonic functions. In particular, we express the transformation rules for the new basis, under any isometry of the sphere.

The development of the usual harmonics $Y_{\ell m}$ into the new basis (and back) allows us to derive new properties for the $Y_{\ell m}$. In particular, this leads to a new relation for the $Y_{\ell m}$, which is a finite version of the well-known integral representation formula. It also provides new development formulae for the Legendre polynomials and for the special Legendre functions.

PACS numbers: 02.30.Gp, 02.40.-k

1. Introduction

For a given value of ℓ , the $2\ell + 1$ spherical harmonics $Y_{\ell m}$ provide a basis for the vector space \mathcal{V}^ℓ of the $2\ell + 1$ dimensional irreducible representation of the group $SO(3)$. The same space is also the eigenspace of the Laplacian on S^2 with eigenvalue $\lambda_\ell = -\ell(\ell + 1)$. Finally, the reunion of all spherical harmonics provides a basis for the functions on the sphere. In some sense, they play for the sphere a role analogous to that of the Fourier modes for the plane, and thus appear very useful for many applications in various fields of mathematics, (classical or quantum) physics, cosmology etc.

This paper presents the construction of another natural basis Φ_j^ℓ , $j = -\ell, \dots, \ell$, for \mathcal{V}^ℓ , which seems to have been ignored in the literature. Each Φ_j^ℓ is defined as (the reduction to the sphere of) a homogeneous harmonic polynomial in \mathbb{R}^3 , which takes the very simple form $(X \cdot N_j)^\ell$, where the dot product extends the Euclidean (scalar) dot product of \mathbb{R}^3 to

its complexification \mathbb{C}^3 , and N_j is a null vector of \mathbb{C}^3 , that we specify below. After defining these functions, we show that they form a basis of \mathcal{V}^ℓ , and we give the explicit transformation formulae between the two bases.

The Φ_j^ℓ have different properties from the $Y_{\ell m}$, which make them more convenient for particular applications. In particular, they are transformed differently by the group $SO(3)$, although in a very simple way. Their main interest comes from their simplicity and that of the calculations involving them. In this respect, we will obtain new formulae concerning the usual spherical harmonics. In particular, we obtain a *finite* version of the usual *integral representation* of the spherical harmonics. And we write new developments of the Legendre polynomials and of the special Legendre functions.

2. Harmonic functions

We parametrize the (unit) sphere with coordinates $\theta = 0, \dots, \pi, \phi = 0, \dots, 2\pi$. An eigenmode (of the Laplacian Δ in S^2) is a (complex) function $f(\theta, \phi)$ on the sphere which verifies $\Delta f = \lambda f$. An eigenvalue is of the form $\lambda_\ell = -\ell(\ell+1)$, $\ell \in \mathbb{N}^+$ and the corresponding eigenfunctions generate the eigen vectorspace \mathcal{V}^ℓ , of dimension $2\ell + 1$, which realizes the irreducible unitary representation of the group $SO(3)$. The usual basis of \mathcal{V}^ℓ is provided by the (complex) *spherical harmonics* $Y_{\ell m}$, $m = -\ell, \dots, \ell$. This allows the development of any (complex) function f on the sphere

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^m f_{\ell m} Y_{\ell m}. \quad (1)$$

The unit sphere S^2 can be isometrically embedded in \mathbb{R}^3 , as the surface $X \cdot X = 1$, when a point of \mathbb{R}^3 is written

$$X = (x, y, z) \quad x = r \cos(\theta) \quad y = r \sin(\theta) \cos(\phi) \quad z = r \sin(\theta) \sin(\phi). \quad (2)$$

Each function $f \in \mathcal{V}^\ell$ may be seen as the restriction of an harmonic polynomial in \mathbb{R}^3 , homogeneous of degree ℓ , with symmetric and traceless coefficients, that we also denote f by an abuse of notation:

$$f(X) = f(x, y, z) = \sum_{m+n+p=\ell} f_{mpq} x^m y^n z^p.$$

2.1. Complex null vectors

A complex vector $Z \equiv (Z^1, Z^2, Z^3)$ is an element of \mathbb{C}^3 . We extend the Euclidean scalar product in \mathbb{R}^3 to the complex (non-Hermitian) inner product $Z \cdot Z' \equiv \sum_i (Z^i)(Z'^i)$, $i = 1, 2, 3$. A *null* vector N is defined as having zero norm $N \cdot N \equiv \sum_i N_i N_i = 0$ (in which case, it may be considered as a point on the isotropic cone in \mathbb{C}^3). It is straightforward, and well known, that polynomials of the form $(X \cdot N)^\ell$, homogeneous of degree ℓ , are harmonic if and only if N is a null vector. This results from

$$\Delta_0 (X \cdot N)^\ell \equiv \sum_i \partial_i \partial_i (X \cdot N)^\ell = \ell \sum_i (N_i N_i) (X \cdot N)^{\ell-1} = 0$$

where Δ_0 is the Laplacian of \mathbb{R}^3 . Thus, the restrictions of such polynomials are in \mathcal{V}^ℓ . Let us define the family of null vectors $N(\alpha)$ with coordinates $(1, i \sin \alpha, i \cos \alpha)$, where the angle α spans the unit circle.

It has been shown by [2] (see also [1]) that, through this family, the $(k+1)^2$ spherical harmonics of degree $\leq k$ of S^{n-1} generate the spherical harmonics of degree $= k$ of S^n . In our case ($n = 3$) this development takes the form

$$[X \cdot N(\alpha)]^\ell = \sum_{m=-\ell}^{\ell} F_{\ell m}(X) e^{im\alpha}. \quad (3)$$

The theorem states that the $F_{\ell m}(X)$ are homogeneous harmonic polynomials of degree ℓ in \mathbb{R}^n . Multiplying the previous relation by $e^{-iM\alpha}$ and integrating over α , we obtain:

$$\begin{aligned} F_{\ell M}(X) &= \int \frac{d\alpha}{2\pi} e^{-iM\alpha} [X \cdot N(\alpha)]^\ell \\ &= \int \frac{d\alpha}{2\pi} e^{-iM\alpha} [\cos \theta + i \sin \theta \sin(\phi + \alpha)]^\ell. \end{aligned} \quad (4)$$

We recognize the well-known integral formula for the spherical harmonics (see, e.g., [3] p 92), which proves that

$$F_{\ell M} = \frac{1}{2\pi B^\ell} Y_{\ell M} \quad B_m^\ell \equiv \frac{1}{4\pi \ell!} \sqrt{\frac{(m+\ell)!(\ell-m)!(2\ell+1)}{\pi}}. \quad (5)$$

Thus, we have the development

$$[X \cdot N(\alpha)]^\ell = \frac{1}{2\pi} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}(X)}{B_m^\ell} e^{im\alpha} \quad (6)$$

which involves the usual (here, complex) spherical harmonics $Y_{\ell m}$. Note that the integral formula allows us to consider the functions $[X \cdot N_\alpha]^\ell$, $\alpha \in S^1$ as *coherent states* for the Hilbert space of the harmonic functions (I thank A Aribe for this remark). In this sense, all these functions form an overcomplete basis for \mathcal{V}^ℓ . We will extract from it a finite basis distinct from that of the $Y_{\ell m}$. (Note that other analogue bases could also be extracted, although less convenient for calculations.)

2.2. A new basis of eigenmodes on the spheres

To find a basis of \mathcal{V}^ℓ in the form of such polynomials, we consider the complex $(2\ell+1)$ th roots of unity, ρ^j , $j = -\ell, \dots, \ell$, with $\rho \equiv e^{i a_\ell}$. Their arguments are the $ja_\ell \equiv j \frac{2\pi}{2\ell+1}$. We will use intensively the property of the roots of unity,

$$\sum_j (\rho^k)^j = (2\ell+1) \delta_k^{\text{Dirac}} \quad (7)$$

the Dirac symbol. With these roots, we associate (in a given frame) the $2\ell+1$ particular null vectors $N_j \equiv N(ja_\ell)$, $j = -\ell, \dots, \ell$. The scalar products are given by $N_j \cdot N_k = 1 - \cos\left(\frac{2\pi(j-k)}{2\ell+1}\right)$. We want to prove that the functions $\Phi_j^\ell : \Phi_j^\ell(X) \equiv (X \cdot N_j)^\ell$ form a basis of \mathcal{V}^ℓ (hereafter we use the same notation for a function in \mathbb{R}^3 and its reduction on the sphere).

Rewriting equation (6) for the particular value $\alpha = ja$ gives

$$\Phi_j^\ell = \frac{1}{2\pi} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}}{B_m^\ell} \rho^{jm}. \quad (8)$$

After multiplication by ρ^{-jM} , the summation over j leads, using (7), to an inversion of the formula, namely

$$Y_{\ell M} = \frac{2\pi B_m^\ell}{2\ell+1} \sum_{j=-\ell}^{\ell} \rho^{-jm} \Phi_j^\ell. \quad (9)$$

This proves that the Φ_j^ℓ , $= -\ell, \ell$ form a basis of \mathcal{V}^ℓ . The last two formulae express the change of bases. As an illustration, we give the first developments in the appendix.

The basis (Φ_j^ℓ) appears not to be orthogonal. However, it is easy to calculate the scalar products

$$\begin{aligned} \langle \Phi_j^\ell, \Phi_{j'}^{\ell'} \rangle &:= \frac{\delta_{\ell\ell'}}{(2\ell+1)^2} \sum_m \rho^{m(j-j')} \frac{1}{(D_{\ell m})^2} \\ &= \frac{4\pi(\ell!)^2}{(2\ell+1)(2\ell)!} \left[2 \cos \left(\frac{(j-j')\pi}{2\ell+1} \right) \right]^{2\ell} \\ &= \frac{4\pi(\ell!)^2 2^\ell}{(2\ell+1)(2\ell)! 2^{2\ell}} [2 - N_j \cdot N_{j'}]^\ell. \end{aligned} \quad (10)$$

2.3. New developments of Legendre polynomials and special Legendre functions

The expansion of (9) with the binomial coefficients reads

$$Y_{\ell m}(\theta, \phi) = \frac{2\pi B_m^\ell}{2\ell+1} \sum_j \rho^{-mj} \sum_{p=0}^{\ell} \binom{\ell}{p} (\cos \theta)^{\ell-p} \left[\frac{\sin \theta}{2} \left(z\rho^j - \frac{1}{z\rho^j} \right) \right]^p$$

where $z \equiv e^{i\phi}$. We develop again, permute the summation symbols, and rearrange the terms: the sum takes the form

$$\sum_{p=0}^{\ell} \binom{\ell}{p} (\cos \theta)^{\ell-p} \left(\frac{\sin \theta}{2} \right)^p \sum_{q=0}^p \binom{p}{q} (-1)^{p-q} z^{2q-p} \sum_j \rho^{j(2q-p-m)}.$$

Because of the property (7), the term in the sum is non-zero only when $2q - p - m = 0$. The term in the development is non-zero only when $p + m$ is even, and $0 \leq q \leq p \leq \ell$. Taking these conditions into account, we rewrite the sum as

$$\sum_{q=\max(0,m)}^{\lfloor \frac{\ell+m}{2} \rfloor} \frac{\ell! 2^{m-2q} (-1)^{q-m}}{(\ell+m-2q)! q! (q-m)!} (\cos \theta)^{\ell+m-2q} (\sin \theta)^{2q-m} \quad (11)$$

where the bracket means entire value.

As expected, $Y_{\ell m}$ is proportional to z^m , and is thus an eigenfunction of the rotation operator P_x . It results in the new development formula for the associated Legendre functions (defined as $P_\ell^m(x) \equiv (1-x^2)^{m/2} \frac{d^m P_\ell(x)}{dx^m}$):

$$P_m^\ell(x) = \sum_{q=\max(0,m)}^{\lfloor \frac{\ell+m}{2} \rfloor} \frac{(m+\ell)! (-1)^{q-m} 2^{m-2q}}{(\ell+m-2q)! q! (q-m)!} (1-x^2)^{q-m/2} x^{\ell+m-2q}. \quad (12)$$

We can specify to the Legendre polynomials, by putting $m = 0$. We obtain

$$P^\ell(x) = \ell! \sum_{q=0}^{\lfloor \frac{\ell}{2} \rfloor} \frac{2^{2q}}{(\ell-2q)! (q!)^2} (x^2-1)^q x^{\ell-2q}. \quad (13)$$

2.4. The integral representation becomes finite

The integral representation of the spherical harmonics is the well known ([3] p 92):

$$Y_{\ell m}(\theta, \phi) = B_m^\ell \int_{-\phi}^{2\pi-\phi} d\alpha e^{-im\alpha} [\cos \theta + i \sin \theta \sin(\phi + \alpha)]^\ell \quad (14)$$

with $B_m^\ell \equiv \frac{1}{4\pi\ell!} \sqrt{\frac{(m+\ell)!(\ell-m)!(2\ell+1)}{\pi}}$. The explicit development of (9) provides a *finite version* of the latter as

$$Y_{\ell m}(\theta, \phi) = \frac{2\pi B_{\ell m}}{2\ell+1} \sum_{j=-\ell}^{\ell} e^{-imj\alpha} [\cos\theta + i \sin\theta \sin(\phi + j\alpha)]^\ell \quad \alpha \equiv \frac{2\pi}{2\ell+1}. \quad (15)$$

2.5. Group action

The vector space \mathcal{V}^ℓ forms a $(2\ell+1)$ dimensional irreducible unitary representation of $SO(3)$, T , whose action is defined through

$$T : SO(3) \ni g \mapsto T_g : f \mapsto T_g f : T_g f(x) \equiv f(gx). \quad (16)$$

This action is completely defined by the transformation laws for a basis of \mathcal{V}^ℓ . For instance, the usual spherical harmonics have the property that they transform very simply (a multiplication by a complex number) under a selected $SO(2)$ subgroup of $SO(3)$: they have been precisely chosen as eigenfunctions of the angular momentum operator, in addition to the Laplacian (which is the Casimir operator of the group).

The functions of the new basis transform in a different way under the group: developing the transformed function in the basis, we can write

$$T_g : \Phi_j^\ell \mapsto T_g \Phi_j^\ell = \sum_k G_j^{(\ell)k} [g] \Phi_k^\ell \quad \forall g \in SO(3). \quad (17)$$

On the other hand, by definition of the representation, $T_g \Phi_j^\ell(X) = \Phi_j^\ell(gX)$. This leads to the relation

$$(gX \cdot N_j)^\ell = \sum_k G_j^{(\ell)k} [g] (X \cdot N_k)^\ell. \quad (18)$$

To go further, we define the two vectors $\alpha \equiv (0, 1, -i)$ and $\beta \equiv (1, 0, 0)$, such that $\alpha \cdot N_j = \rho^j$ and $\beta \cdot N_j = 1$, and we apply the above formula to the vector $X = \alpha + t\beta$. Developing and identifying the powers of t , we obtain

$$(g\alpha \cdot N_j)^p (g\beta \cdot N_j)^{\ell-p} = \sum_k G_j^{(\ell)k} [g] (\alpha \cdot N_k)^p \quad \forall p = 0, \dots, \ell. \quad (19)$$

The solution provides the explicit form of the coefficients

$$G_j^{(\ell)k} [g] = \frac{1}{2\ell+1} \sum_n \rho^{-kn} (g\alpha \cdot N_j)^n (g\beta \cdot N_j)^{\ell-n}$$

for an arbitrary $g \in SO(3)$. This provides the transformation properties of any function developed in the basis.

If we select a subgroup $SO(2)$ in $SO(3)$, there is a basis (in which the pole $[1, 0, 0]$ remains fixed) where its elements are given by the matrices

$$g_\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \quad \psi \in [0, \dots, 2\pi].$$

This implies $g_\psi \alpha \cdot N_j = w\rho^j$, $g_\psi \beta \cdot N_j = 1$, with $w \equiv e^{i\psi}$, and thus

$$G_j^{(\ell)k} = \frac{1}{2\ell+1} \sum_{n=-\ell}^{\ell} w^n \rho^{n(j-k)}. \quad (20)$$

Given the correspondence (8) and (9), this is equivalent to the well-known formula describing the rotation properties of the spherical harmonics, $Y_{\ell m} \mapsto w^m Y_{\ell m}$.

Note that in the case $w = \rho^K$, a $(2\ell+1)$ root of unity, the matrix becomes diagonal: $G_j^{(\ell)k} = \frac{1}{2\ell+1} \sum_{n=-\ell}^{\ell} \rho^{nK} \rho^{n(j-k)} = \delta_{K+j-k}^{\text{Dirac}}$ (a remark due to A Aribe).

3. Conclusion

We have presented a new basis for the eigenfunctions of the Laplacian on the sphere, distinct from the usual basis of spherical harmonics $Y_{\ell m}$, and with different properties. Its very simple expression allows easy calculations. In particular, we gave a general formula which gives their transformation properties under the isometry group of the sphere, $SO(3)$. This led, using the transformation formulae, to easy calculations of the transformation properties of the usual spherical harmonics, not only for a privileged group $SO(2)$ of $SO(3)$. In addition, new formulae were derived for the $Y_{\ell m}$, as well as for the Legendre polynomials and the Legendre special functions, of great potential use for calculations with harmonic functions.

Subsequent work will generalize this construction to S^3 . Besides the intrinsic interest, this will provide an explicit formulation of the transformation properties of the harmonic functions on S^3 under the isometry group $SO(4)$. In turn, this will allow us to select those functions which remain invariant under specified elements of $SO(4)$. This opens the way to calculate the eigenfunctions (of the Laplacian) on any spherical space S^3/Γ , which remain presently unknown in general.

Appendix A

We develop the first basis functions in spherical harmonics:

$$\begin{aligned}\Phi_0^1 &= \sqrt{\frac{2\pi}{3}} [Y_{-1}^1 + \sqrt{2}Y_0^1 + Y_1^1] \\ \Phi_{-1}^1 &= \sqrt{\frac{2\pi}{3}} \left[\frac{-1+i\sqrt{3}}{2} Y_{-1}^1 + \sqrt{2}Y_0^1 - \frac{1+i\sqrt{3}}{2} Y_1^1 \right] \\ \Phi_1^1 &= \sqrt{\frac{2\pi}{3}} \left[-\frac{1+i\sqrt{3}}{2} Y_{-1}^1 + \sqrt{2}Y_0^1 + \frac{-1+i\sqrt{3}}{2} Y_1^1 \right] \\ \Phi_0^2 &= \frac{2\sqrt{\pi}}{\sqrt{5}} \left[\frac{Y_{-2}^1}{\sqrt{6}} + Y_{-1}^1 \sqrt{\frac{2}{3}} + Y_0^1 + Y_1^1 \sqrt{\frac{2}{3}} + \frac{Y_2^1}{\sqrt{6}} \right] \\ \Phi_{-1}^2 &= \frac{2\sqrt{\pi}}{\sqrt{5}} \left[\frac{e^{\frac{4i\pi}{5}} Y_{-2}^1}{\sqrt{6}} + e^{\frac{2i\pi}{5}} Y_{-1}^1 \sqrt{\frac{2}{3}} + Y_0^1 + e^{-\frac{2i\pi}{5}} Y_1^1 \sqrt{\frac{2}{3}} + \frac{e^{-\frac{4i\pi}{5}} Y_2^1}{\sqrt{6}} \right] \\ \Phi_1^2 &= \frac{2\sqrt{\pi}}{\sqrt{5}} \left[\frac{e^{-\frac{4i\pi}{5}} Y_{-2}^1}{\sqrt{6}} + e^{-\frac{2i\pi}{5}} Y_{-1}^1 \sqrt{\frac{2}{3}} + Y_0^1 + e^{\frac{2i\pi}{5}} Y_1^1 \sqrt{\frac{2}{3}} + \frac{e^{\frac{4i\pi}{5}} Y_2^1}{\sqrt{6}} \right] \\ \Phi_2^2 &= \frac{2\sqrt{\pi}}{\sqrt{5}} \left[\frac{e^{-\frac{8i\pi}{5}} Y_{-2}^1}{\sqrt{6}} + e^{-\frac{4i\pi}{5}} Y_{-1}^1 \sqrt{\frac{2}{3}} + Y_0^1 + e^{\frac{4i\pi}{5}} Y_1^1 \sqrt{\frac{2}{3}} + \frac{e^{\frac{8i\pi}{5}} Y_2^1}{\sqrt{6}} \right] \\ \Phi_{-2}^2 &= \frac{2\sqrt{\pi}}{\sqrt{5}} \left[\frac{e^{\frac{8i\pi}{5}} Y_{-2}^1}{\sqrt{6}} + e^{\frac{4i\pi}{5}} Y_{-1}^1 \sqrt{\frac{2}{3}} + Y_0^1 + e^{-\frac{4i\pi}{5}} Y_1^1 \sqrt{\frac{2}{3}} + \frac{e^{-\frac{8i\pi}{5}} Y_2^1}{\sqrt{6}} \right].\end{aligned}$$

References

- [1] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F G 1953 *Higher Transcendental Functions* (New York: McGraw-Hill) pp 248–500
- [2] Fryant A 1991 *SIAM J. Math. Anal.* **22** 268–71
- [3] Nikiforov A and Ouvarov V 1983 *Fonctions spéciales de la physique mathématique* (Moscow: Mir)